Topic 2 - Counting and Probability

Review of factorial

Def: For integers $n \geq 0$ define:

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1 \end{cases}$$

Ex: $0! = 1$

$1! = 1 \cdot 0! = 1$

$2! = 2 \cdot 1! = 2 \cdot 1 = 2$

$3! = 3 \cdot 2! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$5! = 5 \cdot 4! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

Ex: We can do stuff like this!

$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= 10 \cdot 9 \cdot 8 \cdot [7!]$

We will do this alot

Basic counting principle

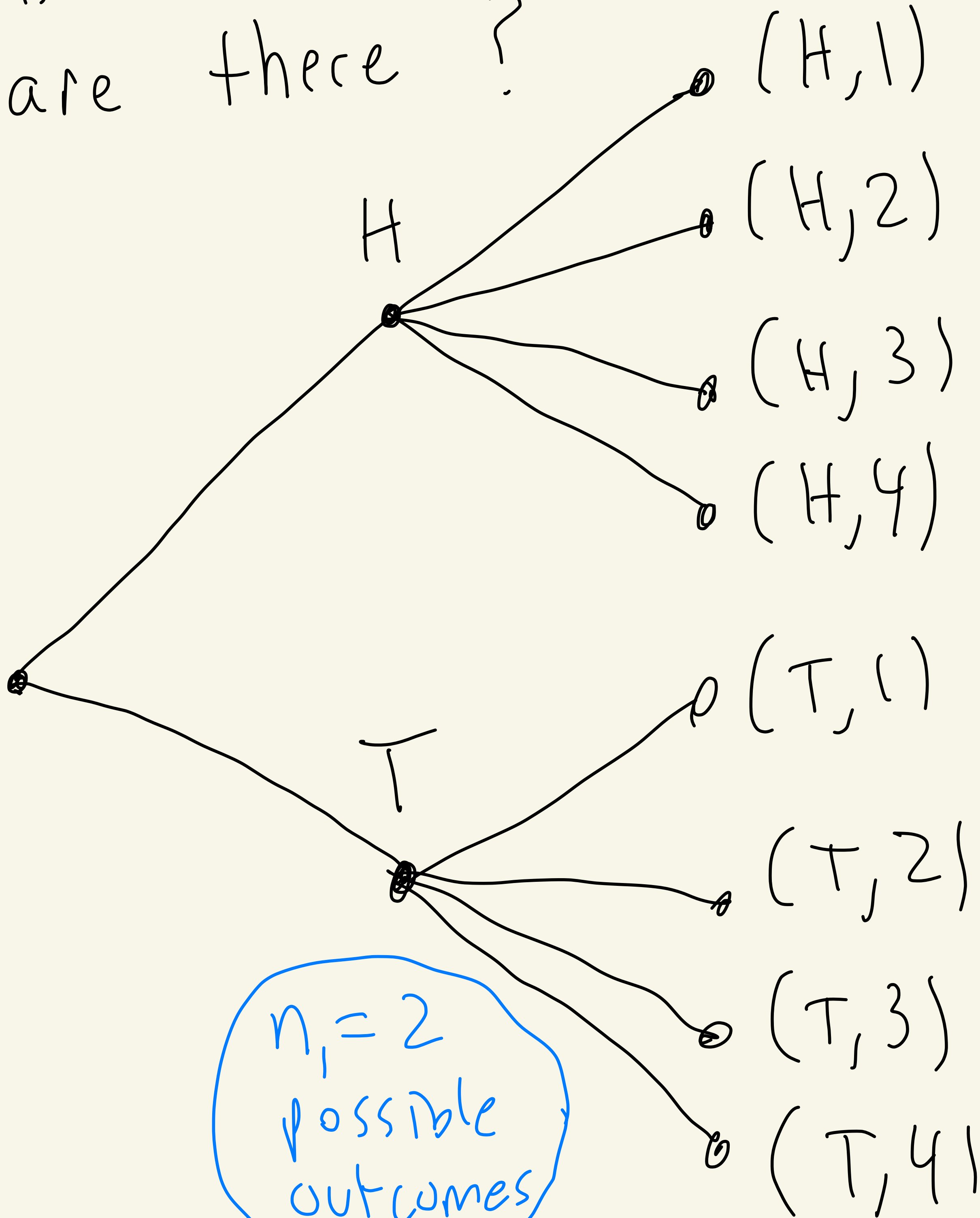
If r experiments are performed in a row such that the first experiment may result in n_1 possible outcomes; and if for each of these n_1 possible outcomes there are n_2 possible outcomes for the second experiment; and if for each of the possible outcomes of the first two experiments there are n_3 possible outcomes of the

third experiment; and if, ...,
then there are

$$n_1 \cdot n_2 \cdot n_3 \cdots n_r$$

possible outcomes for the
r experiments.

Ex: Suppose we toss a coin and then roll a 4-sided die. How many possible outcomes are there?



$n_1 \cdot n_2 = 2 \cdot 4 = 8$ overall possible outcomes

$n_1 = 2$ possible outcomes

$n_2 = 4$

Another way to write:

$$\overbrace{\underline{\;H/T\;}}^{\text{means H or T}} \qquad \overbrace{\underline{\;1,2,3,4\;}}^{\text{means 1, 2, 3, or 4}}$$

$$\underset{\text{possibilities}}{2} \cdot \underset{\text{possibilities}}{4} = 8$$

Ex: In California, a license plate consists of one number (0, 1, 2, 3, 4, 5, 6, 7, 8, or 9) followed by three upper-case letters, followed by three numbers. The only exclusion is that the letters I, O, and Q are not used in spot 2 and spot 4.

Examples are:

$\underline{5}\ \underline{K}\ \underline{A}\ \underline{T}\ \underline{9}\ \underline{9}\ \underline{2}$

$\underline{3}\ \underline{A}\ \underline{Q}\ \underline{A}\ \underline{1}\ \underline{2}\ \underline{3}$

How many possible license plates are there?

#	letter not I, O, Q	letter	letter not I, O, Q	#	#	#
$n_1 = 10$	$n_2 = 26 - 3 = 23$	$n_3 = 26$	$n_4 = 23$	$n_5 = 10$	$n_6 = 10$	$n_7 = 10$

total # of possible license plates is

$$10 \cdot 23 \cdot 26 \cdot 23 \cdot 10 \cdot 10 \cdot 10$$

$$= \boxed{137{,}540{,}000}$$

Birthday Paradox

Suppose there are N people in a classroom. What are the odds (probability) that there are at least two people with the same birthday? (This means month & day, not necessarily year. Such as at least two people born on 9/4)

Assumptions:

(1) We will assume that no one has a Feb 29

leap year birthday.

(2) We will assume that each day is equally likely

(3) Assume $N \leq 365$ because if $N > 365$ then the probability is 100%

Let's figure out the sample space.

What if $N = 3$?

$$S = \left\{ (\text{date } 1, \text{date } 2, \text{date } 3) \;\middle|\; \begin{array}{c}\text{date } i \\ \text{is a} \\ \text{calendar} \\ \text{day}\end{array} \right\}$$

$$= \{ (\underbrace{\text{April 1}}_{\text{student 1}}, \underbrace{\text{May 10}}_{\text{student 2}}, \underbrace{\text{Feb 3}}_{\text{student 3}}),$$

$$(\underbrace{\text{Jan 17}}_{\text{student 1}}, \underbrace{\text{Oct 5}}_{\text{student 2}}, \underbrace{\text{July 4}}_{\text{student 3}}),$$

ex of two with same bday

$$(\underbrace{\text{Jan 15}}_{\text{student 1}}, \underbrace{\text{Oct 3}}_{\text{student 2}}, \underbrace{\text{Jan 15}}_{\text{student 3}}),$$

$$\ldots \}$$

Then, $|S| = 365 \cdot 365 \cdot 365$
$= (365)^3$

For general N, the size of the sample space is $(365)^N$

$$\underset{\text{student 1}}{\underline{\text{365 possibilities}}} \cdot \underset{\text{student 2}}{\underline{\text{365 possibilities}}} \cdots \underset{\text{student N}}{\underline{\text{365 possibilities}}}$$

Let E be the event that there are at least two people with the same birthday.

This is too hard to count.

So instead we count $\overline{E}$ which is the event that no one has the same birthday.

Let's count the size of $\overline{E}$

$$\underset{\text{student 1}}{\underline{\text{365 possibilities}}} \quad \underset{\text{student 2}}{\underline{\text{364 possibilities}}} \quad \underset{\text{student 3}}{\underline{\text{363 possibilities}}} \cdots \underset{\text{student N}}{\underline{365-(N-1)}}$$

Cant have same bday as student 1

Cant have same bday as student 1 or student 2

cant have same bday as previous N−1 students

So,

$$|\overline{E}| = \underbrace{365 \cdot 364 \cdot 363 \cdots (365-(N-1))}_{\frac{365!}{(365-N)!}}$$

will get to this later

Thus,

thm last week

$$\underbrace{P(E)}_{\text{our goal}} = 1 - P(\overline{E})$$

$$= 1 - \frac{|\overline{E}|}{|S|}$$

assumed every day equally likely

$$= 1 - \frac{365 \cdot 364 \cdot 363 \cdots (365-N+1)}{(365)^N}$$

When $N=3$ you get

$$P(E) = 1 - \frac{365 \cdot 364 \cdot 363}{(365)^3}$$

$$\approx 0.0082 \approx 0.82\%$$

N	$P(E)$
1	0%
2	0.274%
3	0.82%
4	1.64%

5	2,71 %
⋮	⋮
10	11.7 %
⋮	
18	34.7 %
⋮	
24	53.83 %
⋮	
40	89.12 %
⋮	
50	97.04 %

See the full table on the next page

N	probability that at least two people from a group of N people have the same birthday
1	0 %
2	0.274 %
3	0,82 %
4	1,64 %
5	2.71 %
6	4,05 %
7	5.62 %
8	7,43 %
9	9,46 %
10	11,69 %
11	14,11 %
12	16,7 %
13	19,44 %
14	22,31 %
15	25.29 %
16	28.36 %
17	31,5 %
18	34,69 %
19	37,91 %
20	41.14 %
21	44,37 %
22	47.57 %
23	50,73 %
24	53,83 %
25	56.87 %
26	59.82 %
27	62.69 %
28	65.45 %
29	68,1 %
30	70.63 %
31	73.05 %
32	75.33 %
33	77.5 %
34	79.53 %
35	81,44 %
36	83.22 %
37	84.87 %
38	86,41 %
39	87,82 %
40	89,12 %
41	90.32 %
42	91.4 %
43	92.39 %
44	93,29 %
45	94.1 %
46	94,83 %
47	95,48 %
48	96.06 %
49	96,58 %
50	97,04 %

Permutations

Suppose you have n objects. A permutation of those n objects is an ordered list of the n objects.

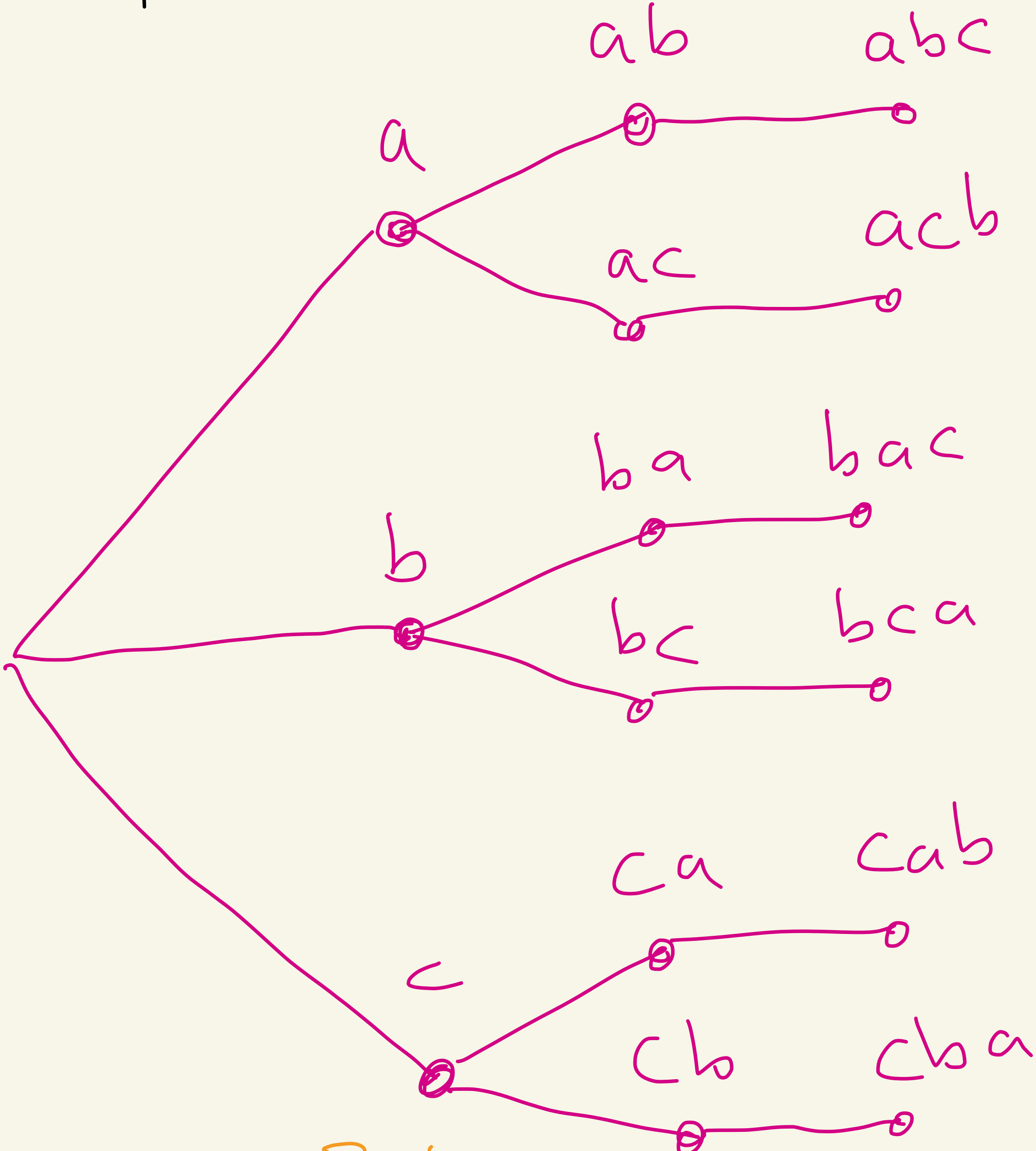
Ex: What are all the permutations of a, b, c ?

permutations:		another way:
a b c	⟵	(a, b, c)
a c b	⟵	(a, c, b)
b a c	⟵	(b, a, c)
b c a	⟵	(b, c, a)
c a b	⟵	(c, a, b)
c b a	⟵	(c, b, a)

simpler way

math way to make order matter

6 possible permutations-



3 choices · 2 choices · 1 choice

$\underset{\text{choices}}{\underline{3}} \cdot \underset{\text{choices}}{\underline{2}} \cdot \underset{\text{choice}}{\underline{1}}$

$3 \cdot 2 \cdot 1 = 3!$ possibilities

In general, there are $n!$ permutations of n objects

$\underline{n}\ \underline{n-1}\ \underline{n-2} \cdots \underline{1}$

Ex: In how many ways can 5 people be seated in a row of 5 seats?

Example seatings

seat 1	seat 2	seat 3	seat 4	seat 5
M	B	C	S	D
M	B	C	D	S
D	M	C	B	S

5 people

Ben
Chris
Derick
Monica
Shaq

Answer:

possibilities:

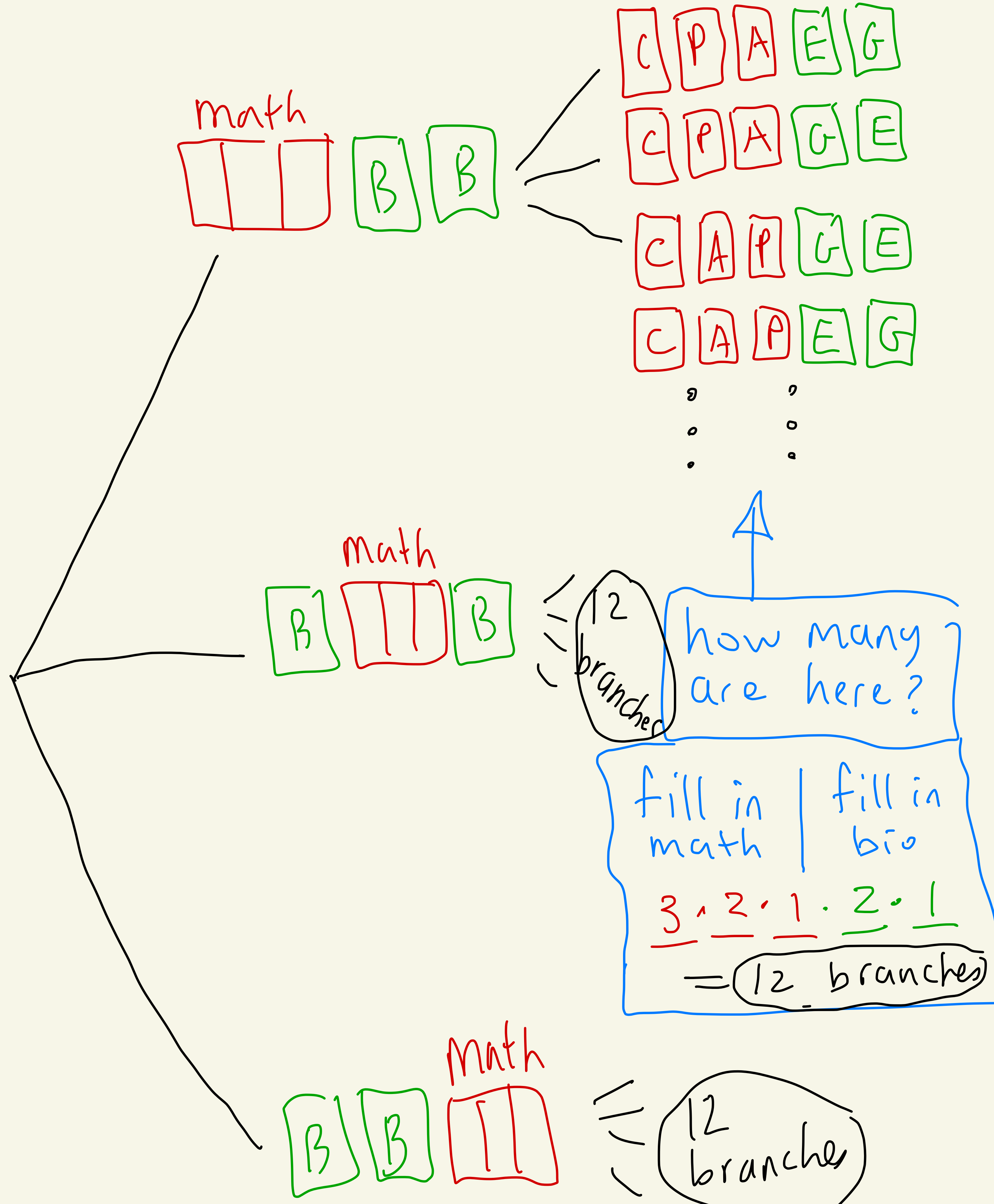
5 choices	· 4 choices	· 3 choices	· 2 choices	· 1 choice
seat 1	seat 2	seat 3	seat 4	seat 5

$$= 5! = \boxed{120}$$

Ex: Suppose we have 3 math books and 2 biology books. How many ways can we put the books on a shelf so that the math books are next to each other?

Ex:

Math	Bio
Calculus	Evolution
Probability	Genetics
Algebra	

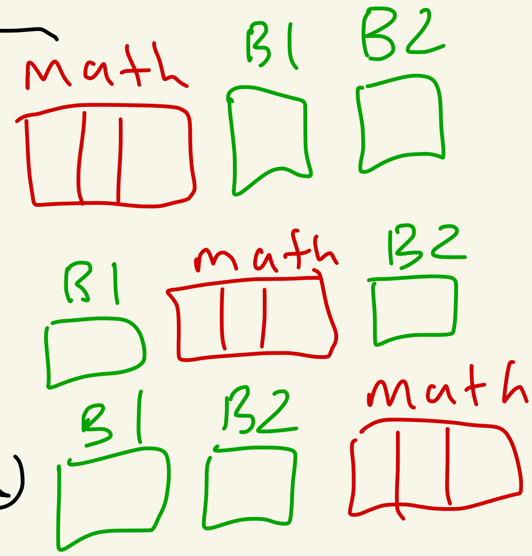


36 total arrangements

Another way to count:

$\boxed{\text{Step 1}}$

Pick one of these



$\boxed{\text{3 possibilities in step 1}}$

$\boxed{\text{Step 2}}$ Fill in the books

$$\underbrace{3\cdot2\cdot1}_{\text{Fill in math}} \cdot \underbrace{2\cdot1}_{\text{Fill in bio}} = \boxed{\text{12 possibilities in step 2}}$$

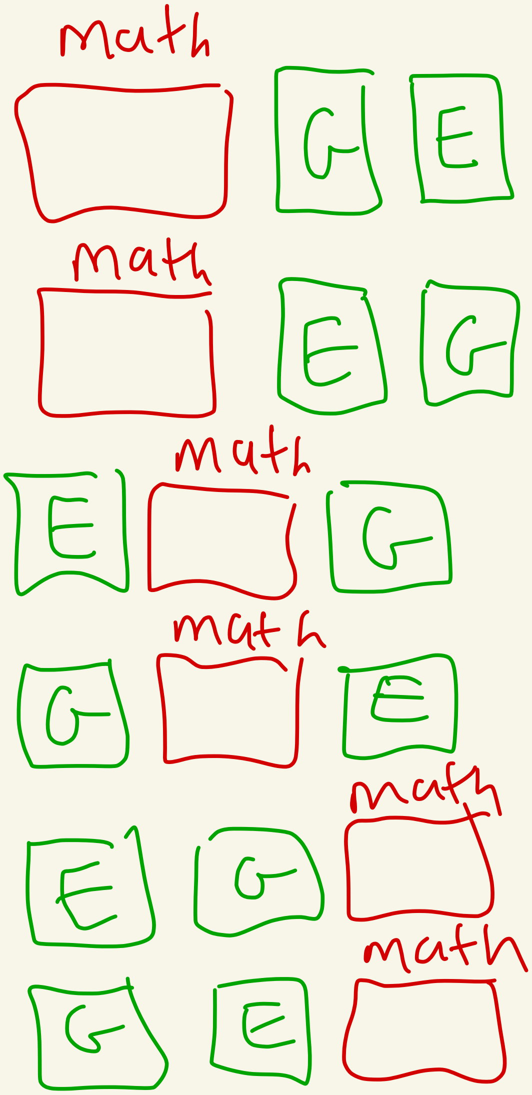
$$\text{Total} = \underbrace{3}_{\text{step 1}} \cdot \underbrace{12}_{\text{step 2}} = 36$$

Another way:

Think of math as a unit and two bio as seperate.

So, 3 objects.

Step 1: Order the 3 objects



$$3! = 6$$

Step 2: Fill in math chunk

math

3	2	1

3! ways to do this

$3! = 6$

Answer $= \underset{\boxed{\text{Step 1}}}{6} \cdot \underset{\boxed{\text{Step 2}}}{6} = 36$

Suppose we have n objects where n_1 of them are alike (ie the same or indistinguishable), n_2 of them are alike, $\ldots$, n_r are alike

where $n = n_1 + n_2 + \cdots + n_r$

Then there are

$$\frac{n!}{n_1!\, n_2! \cdots n_r!}$$

permutations of these objects

Ex: How many permutations are there of the letters a, a, b, c ?

permutations

aabc
aacb
abac
acab
abca
acba
baac
caab
baca
caba
bcaa
cbaa

12 permutations

formula way

$n_1 = 2$ ← # a's

$n_2 = 1$ ← # b's

$n_3 = 1$ ← # c's

$n = 4$ ← total

permutations:

$$\frac{4!}{2!\,1!\,1!} = \frac{24}{2} = \boxed{12}$$

Combinations

Consider a set of size n.
The number of subsets of size r where $0 \le r \le n$ is

$$\underbrace{\binom{n}{r}}_{\text{read: "n choose r"}} = \frac{n!}{r!(n-r)!}$$

look at Spring 22 notes for formula derivation

This is the same as the # of ways that r objects can be selected/chosen from n objects where order doesn't matter proof: There are

$$\underline{n} \cdot \underline{(n-1)} \cdot \underline{(n-2)} \cdot \cdots \cdot \underline{(n-r+1)}$$

ways to write all permutations of r of the n objects. Then divide by $r!$ to remove all the double counting.

This gives

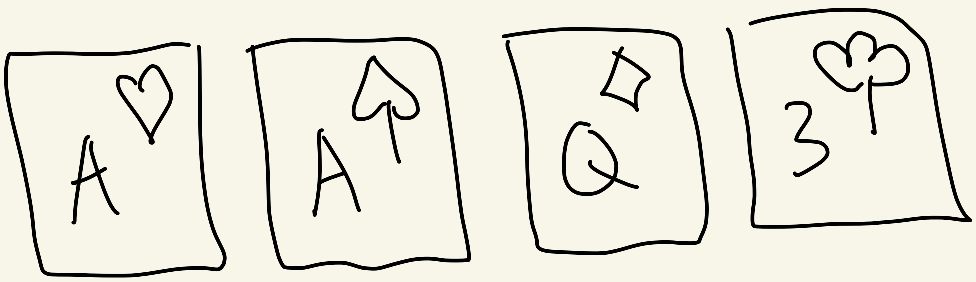
$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \cdot \frac{(n-r)!}{(n-r)!}$$

$$= \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

ways to pick r of the n objects where order doesn't matter. $\blacksquare$

Ex:

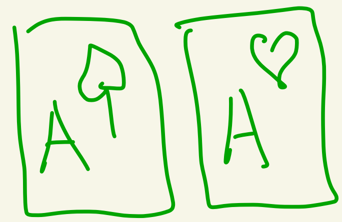
Suppose a dealer has the following cards:



How many ways can the dealer deal you two cards from these four?

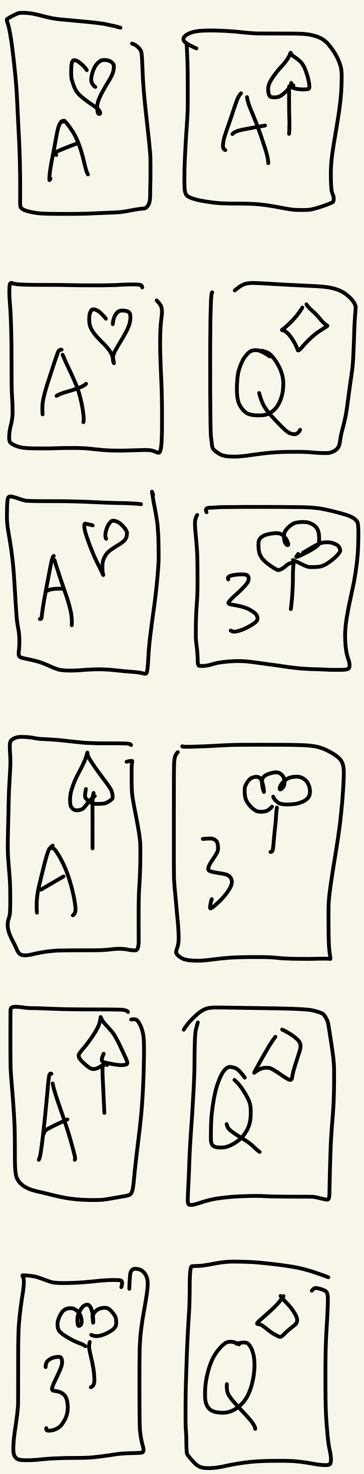
Ex:

← same as



order doesnt matter

possibilities



6 possibilities

formula way

$$\boxed{4 \text{ choose } 2}$$

$$\binom{4}{2} = \frac{4!}{2!(4-2)!}$$

$$= \frac{4!}{2!\,2!}$$

$$= \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 2}$$

$$= \frac{24}{4}$$

$$= 6$$

Ex: A dealer has a standard 52-card deck. They deal you 5 cards. How many possible hands can you get?

Ex hand: 10♣ J♣ Q♣ K♣ A♣

Royal Flush!

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Answer:

$$\binom{52}{5} = \frac{52!}{5!(52-5)!}$$

$$= \frac{52!}{5!\,47!}$$

$$= \frac{\overset{13}{\cancel{52}}\cdot\overset{17}{\cancel{51}}\cdot\overset{10}{\cancel{50}}\cdot 49\cdot\overset{24}{\cancel{48}}\cdot\cancel{47!}}{(\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1)\,\cancel{47!}}$$

$$= 13\cdot 17\cdot 10\cdot 49\cdot 24$$

$$= \boxed{2{,}598{,}960}$$

Website — Show CA Superlotto Plus website

Video — Show CA Superlotto Plus selection of #s video

See website for the links

CA Superlotto Plus

A ticket consists of:

- 5 "lucky" numbers chosen from 1–47
- 1 "mega" number chosen from 1–27

- No repeats in the lucky numbers. But the mega number can be the same as a lucky number.
- Order of lucky #s doesn't matter. It's always in numerical order on the ticket.

Example tickets:

$\underbrace{7 \quad 13 \quad 18 \quad 23 \quad 40}_{\text{lucky \#s}} \quad \underbrace{23}_{\text{mega\#}}$ ← ex ticket #1

$$\underbrace{1 \quad 2 \quad 3 \quad 4 \quad 5}_{\text{lucky \#s}} \qquad \underbrace{6}_{\text{mega \#}}$$

← ex ticket #2

How many possible tickets are there?

If you want to think of a sample space of all possible tickets:

$$S = \{ \underbrace{(\{7,13,18,23,40\}, 23)}_{\text{ticket 1}},$$
$$\underbrace{(\{1,2,3,4,5\}, 6)}_{\text{ticket 2}}, \ldots \}$$

↑ lots more

How many possible tickets?

$$\underbrace{\binom{47}{5}}_{\substack{\text{\# of ways} \\ \text{to pick 5} \\ \text{lucky \#s} \\ \text{from 1-47}}} \cdot \underbrace{\binom{27}{1}}_{\substack{\text{\# ways} \\ \text{to pick} \\ \text{1 mega \#} \\ \text{from 1-27}}}$$

$$= \frac{47!}{5!(47-5)!} \cdot 27$$

$$= \frac{47!}{5!\,42!} \cdot 27$$

$$=$$

Fact:

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n \cdot (n-1)!}{(n-1)!} = n$$

$$8! = 8[7!]$$

That is,

$$\binom{n}{1} = n$$

$$= \frac{47 \cdot \cancel{46}^{\,23} \cdot \cancel{45}^{\,\cancel{9}\,3} \cdot \cancel{44}^{\,11} \cdot 43 \cdot \cancel{(42!)}}{(\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1)\cancel{(42!)}} \cdot 27$$

$$= 47 \cdot 23 \cdot 3 \cdot 11 \cdot 43 \cdot 27$$

$$= \boxed{41{,}416{,}353 \text{ possible tickets}}$$

Q: What is the probability that if you buy one ticket you will get the 5 lucky #s correct and the mega # correct?

A: $$\frac{1}{41{,}416{,}353} \approx 0.00000002414\ldots$$

$$\approx 0.000002414\,\%$$

is the probability

Q: What are the odds of getting exactly 3 of the 5 lucky #s and not the mega #?

#'s drawn by the magical lottery machine

3 12 15 41 42 17

$\underbrace{3 \quad 12 \quad 15 \quad 41 \quad 42}_{\text{lucky \#s}} \quad \underbrace{17}_{\text{mega}}$

How many tickets will get exactly 3 of the 5 lucky #s and not the mega?

you want your ticket in this group

$\boxed{47-5=42}$

$$\underbrace{\binom{5}{3}}_{\text{choose 3 of the 5 winning}} \cdot \underbrace{\binom{42}{2}}_{\text{choose 2 non-winning}} \cdot \underbrace{\binom{26}{1}}_{\text{not picking winning}} =$$

	lucky #s	lucky #s	mega #
Ex:	3, 15, 42	1, 7	1
	12, 41, 42	43, 45	12
	⋮	⋮	⋮

$$\frac{5!}{3!(5-3)!} \cdot \frac{42!}{2!(42-2)!} \cdot 26$$

$$= \frac{5!}{3!\,2!} \cdot \frac{42!}{2!\,40!} \cdot 26$$

$$= \frac{120}{(6)(2)} \cdot \frac{42 \cdot 41 \cdot \cancel{(40!)}}{(2)\cancel{(40!)}} \cdot 26$$

$$= (10)(861)(26)$$

$$= \boxed{223{,}860 \text{ tickets}}$$

$$\text{Probability} = \frac{223,860}{41,416,353}$$

$$\approx 0.00540511\ldots$$

$$\approx \boxed{0.540511\ \%}$$

lottery website says the probability is

$$\frac{1}{185} \approx 0.00540541\ldots$$

Ex: Suppose five 6-sided dice are rolled. What is the probability that exactly two of the dice have 6's showing?

Ex:

6	1	2	6	4
die 1	die 2	die 3	die 4	die 5

Sample space size:

$$\frac{6 \text{ possibilities}}{\text{die 1}} \quad \frac{6 \text{ possibilities}}{\text{die 2}} \quad \frac{6 \text{ possibilities}}{\text{die 3}} \quad \frac{6 \text{ possibilities}}{\text{die 4}} \quad \frac{6 \text{ possibilities}}{\text{die 5}}$$

$$= 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 = 6^5$$

$$= \boxed{7,776}$$

How many rolls have exactly two 6's?

Step 1: Choose two of the dice to get the two 6's.

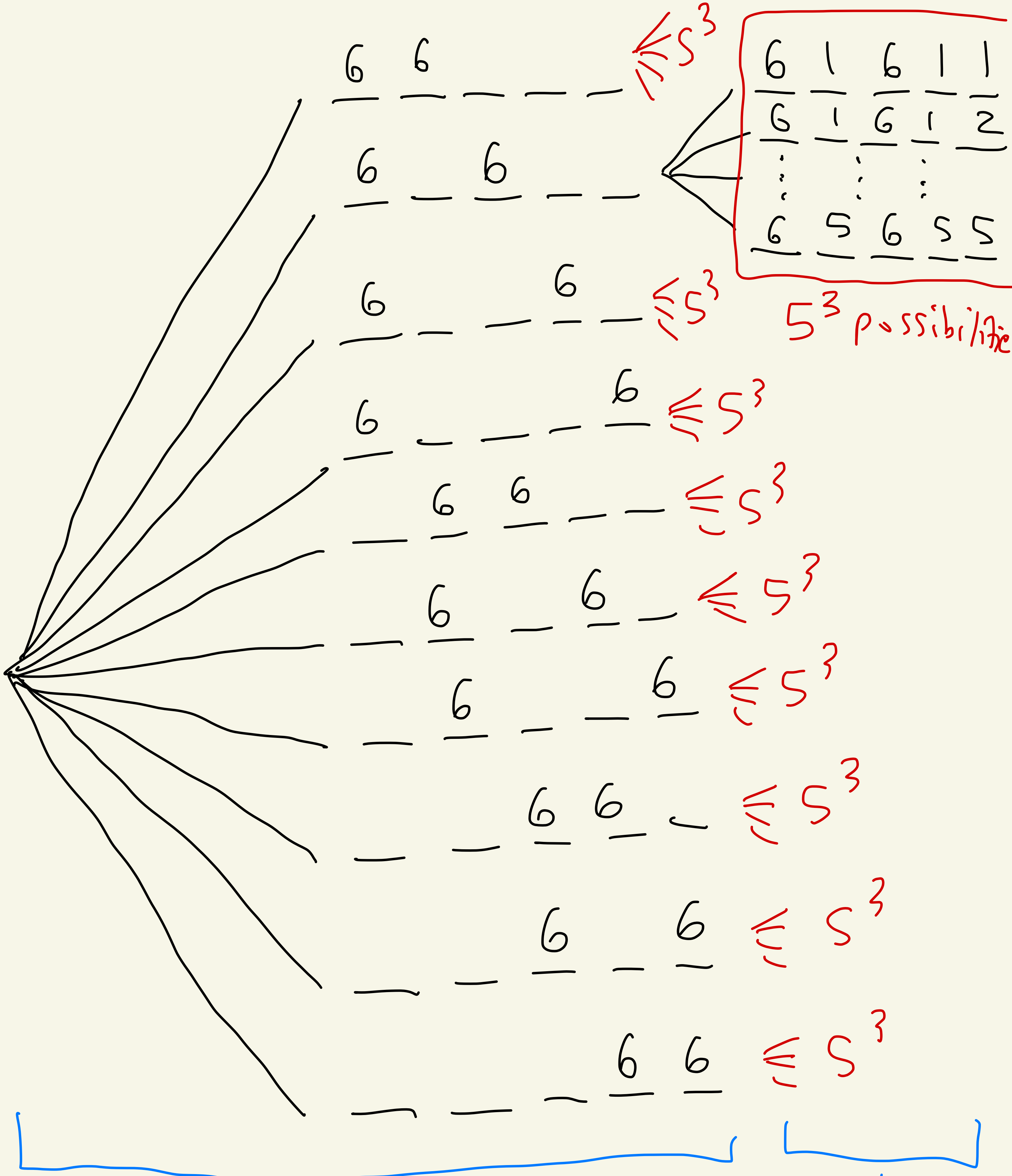
There are $\binom{5}{2} = 10$ ways to do this.

6		6		
die 1	die 2	die 3	die 4	die 5

Step 2: Fill in the non-6's.

6	5 choices	6	5 choices	5 choices
die 1	die 2	die 3	die 4	die 5

There are 5^3 ways to do this.



Step 1: $\binom{5}{2} = 10$ possibilities

Step 2: 5^3

Answer:

$$\frac{10 \cdot 5^3}{7,776}$$

$$\approx 0.16075...$$

$$\approx 16\%$$

chance you get exactly two 6's.

HW 2 problem

1 Suppose you are dealt 2 cards from a standard 52-card deck.

(a) What's the probability that both cards are aces?

(b) What's the probability both cards have the same face value (or rank)?

HW also has part (c) blackjack question

The sample space size is the total # of 2-card hands.

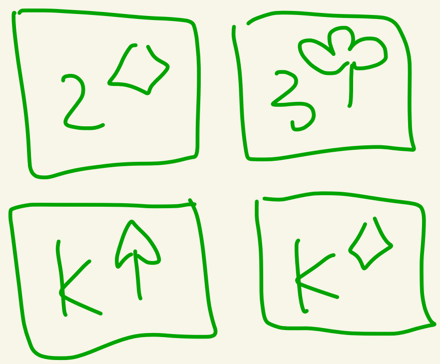
It is

$$\binom{52}{2} = \frac{52!}{2!\,(52-2)!} = \frac{52\cdot 51\cdot \cancel{(50!)}}{2\cdot \cancel{(50!)}}$$

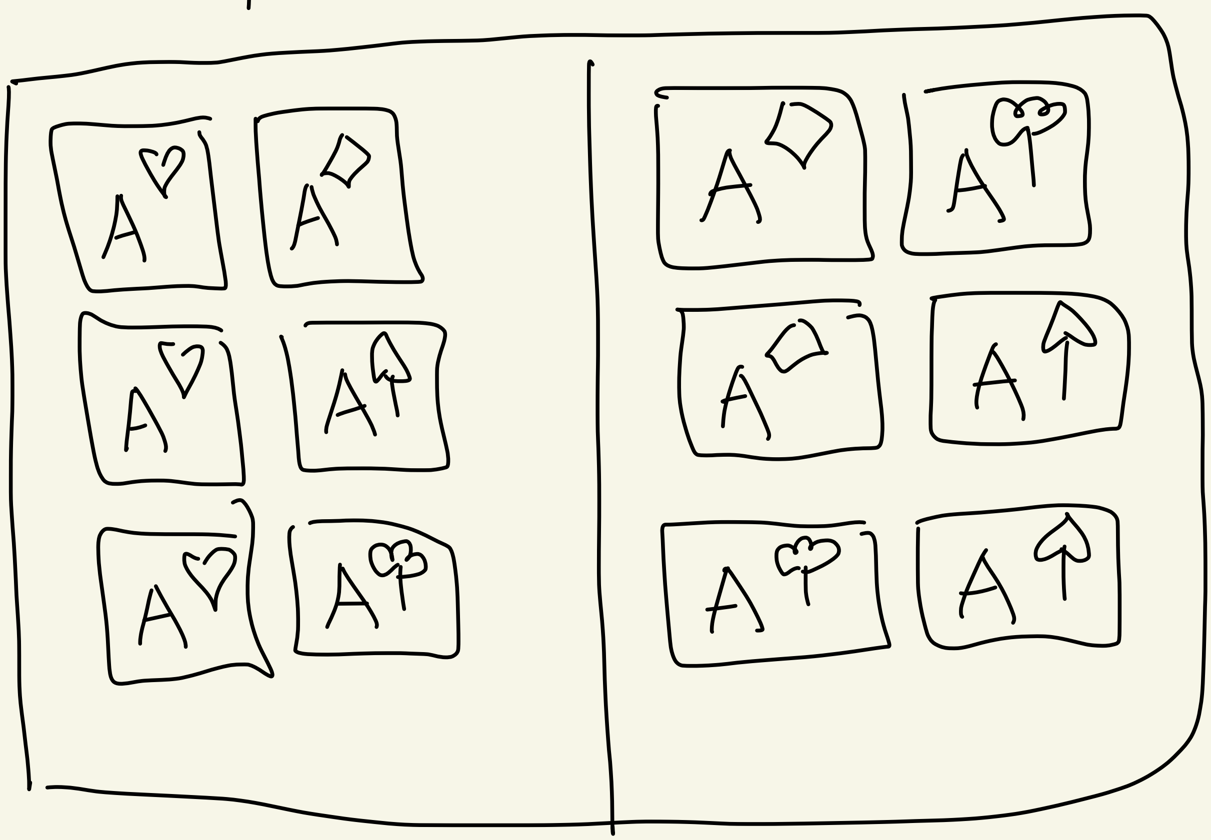
$$= 26\cdot 51$$

$$= \boxed{1326}$$

Ex:



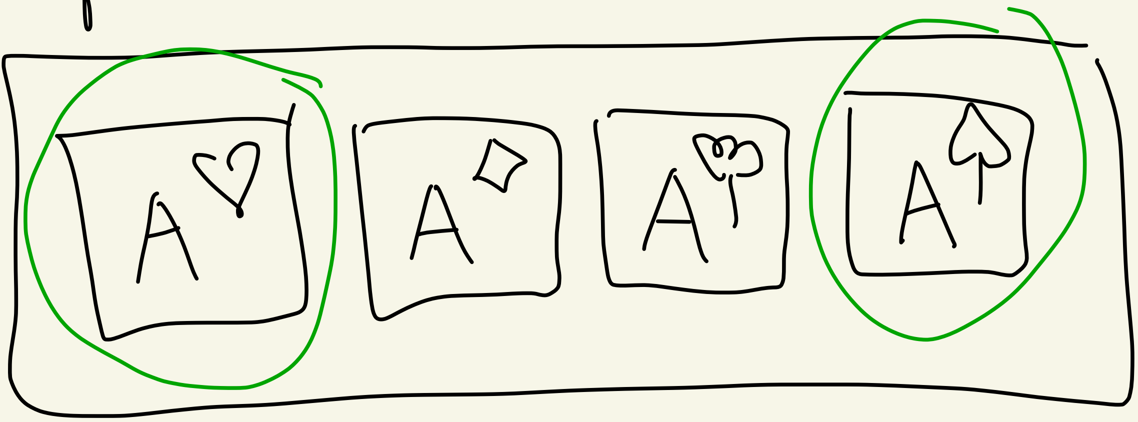
(a) How many hands have two aces?



6 hands

or use choosing.

pick 2 from:



$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 2} = 6$$

the probability of getting two aces is

$$\frac{6}{1326} = \frac{1}{221} \approx 0.00452\ldots$$

$$\approx 0.452\%$$

(b) Need to count the # of hands with both cards same face value.

Step 1: Pick the face value

A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K

possibilities in step 1: $\binom{13}{1} = 13$

Ex: 7 7

Step 2: Pick two suits

♠, ♣, ♥, ♦

$\boxed{\# \text{ possibilities in step 2: } \binom{4}{2} = 6}$

Ex: $7^{\clubsuit}$ $7^{\diamondsuit}$

# of hands with same face value on both cards is

$$\underbrace{13}_{\text{step 1}} \cdot \underbrace{6}_{\text{step 2}} = \boxed{78}$$

Probability is $\dfrac{78}{1326} = \dfrac{1}{17}$

$\approx 0.0588..$

$\approx 5.88\%$

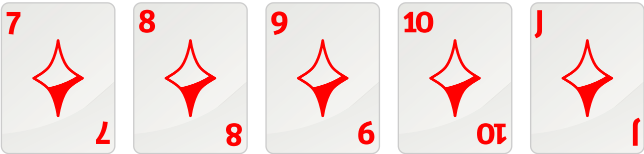
POKER cheatsheet

selectabet.net

In poker, certain combinations of cards, or hands, outrank other hands, based on the frequency with which these combinations appear. The player with the best poker hand at the showdown wins the pot.

ROYAL FLUSH

A straight from a ten to an ace and all five cards of the same suit. In poker suit does not matter and pots are split between equally strong hands.

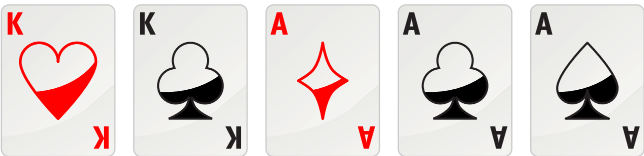


STRAIGHT FLUSH

Any straight with all five cards of the same suit.

FOUR OF A KIND

Any four cards of the same rank. If two players share the same Four of a Kind, the fifth card will decide who wins the pot, the bigger card the better.



FULL HOUSE

Any three cards of the same rank together with any two cards of the same rank. Our example shows "Aces full of Kings" and it is a bigger full house than "Kings full of Aces."

FLUSH

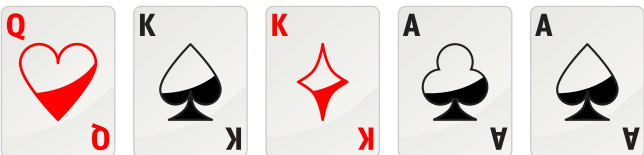
Any five cards of the same suit which are not consecutive. The highest card of the five makes out the rank of the flush. Our example shows an Ace-high flush.

STRAIGHT

Any five consecutive cards of different suits. The ace count as either a high or a low card. Our example shows a Five-high straight, which is the lowest possible straight.

THREE OF A KIND

Any three cards of the same rank. Our example shows three of a kind in Aces with a King and a Queen as side cards, which is the best possible three of a kind.



TWO PAIR

Any two cards of the same rank together with another two cards of the same rank. Our example shows the best possible two-pair, Aces and Kings. The highest pair of the two make out the rank of the two-pair.

ONE PAIR

Any two cards of the same rank. Our example shows the best possible one-pair hand.

HIGH CARD

Any hand that does not make up any of the above mentioned hands. Our example shows the best possible High-card hand.

Visit **selectabet.net** for free chips at all the best online poker sites

Ex 5-card poker hands:

J♦ 2♣ J♥ 3♠ J♠

3-of-a-kind
3 jacks

2♣ 3♠ 4♥ 2♦ A♦

pair

3♣ 3♠ 2♥ 2♦ A♠

two pair

3♣ 3♠ 2♥ 2♦ 2♠

pair + 3-of-a-kind

full house

2♦ 3♠ 4♣ 5♥ A♠ ← straight

Same as:

A♠ 2♦ 3♠ 4♣ 5♥

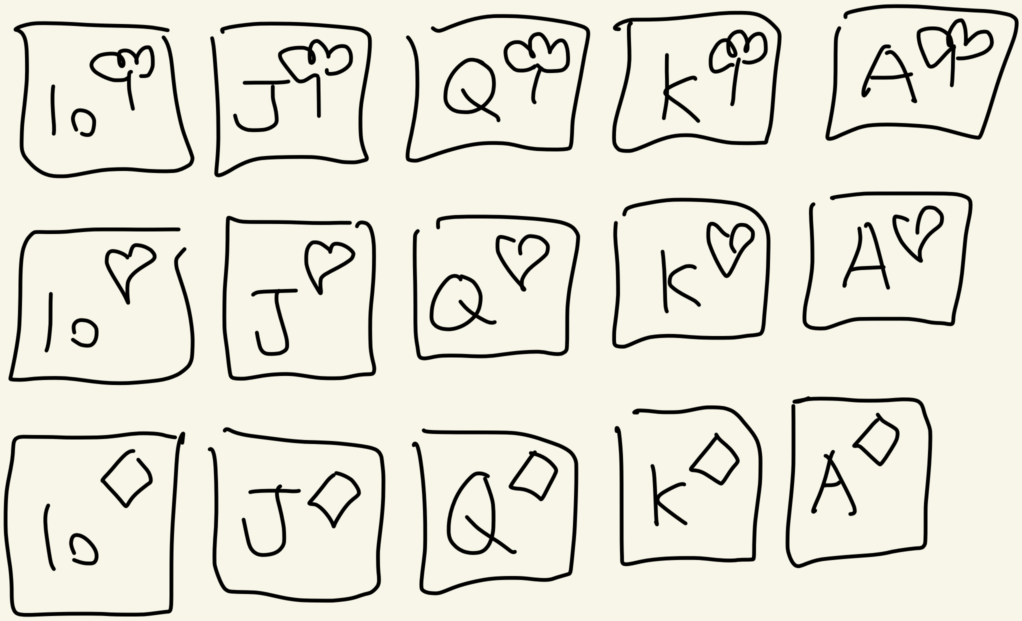
Ex: Suppose you are dealt 5 cards from a standard 52-card deck.
What's the probability that you get a royal flush?

The size of the sample space, ie the total # of possible 5-card poker hands is

$$\binom{52}{5} = 2,598,960$$

How many royal flushes are there?

$\boxed{10\spadesuit}$ $\boxed{J\spadesuit}$ $\boxed{Q\spadesuit}$ $\boxed{K\spadesuit}$ $\boxed{A\spadesuit}$



4 royal flushes

The probability of a royal flush is

$$\frac{4}{2{,}598{,}960} = \frac{1}{649{,}740}$$

$$\approx 0.000001539\ldots$$

$$\approx 0.0001539\,\%$$

Ex: Same setup as above, what's the probability of getting one pair and nothing better?

---

sample space size:

$$\binom{52}{5} = 2,598,960$$

---

we need to count the # of hands that make a pair and nothing better.

A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K $\leftarrow$ $\boxed{\text{face value}}$

♠, ♣, ♥, ♦ $\leftarrow$ $\boxed{\text{suit}}$

Let's enumerate the pairs.

Step 1: Pick a face value for the pair.

A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K

possibilities in step 1: $\binom{13}{1} = 13$

Ex: $\boxed{2}\ \boxed{2}\ \boxed{}\ \boxed{}\ \boxed{}$

Step 2: Pick 2 suits for the pair

♣, ♠, ♢, ♡

Possibilities in step 2: $\binom{4}{2} = 6$

Ex: 2♣ 2♥

Step 3: Pick the other 3 face values. They can't be the same as step 1, and you can't pick any duplicates.

A, ~~2~~, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K

possibilities in step 3: $\binom{12}{3} = \frac{12!}{3!(12-3)!} = 220$

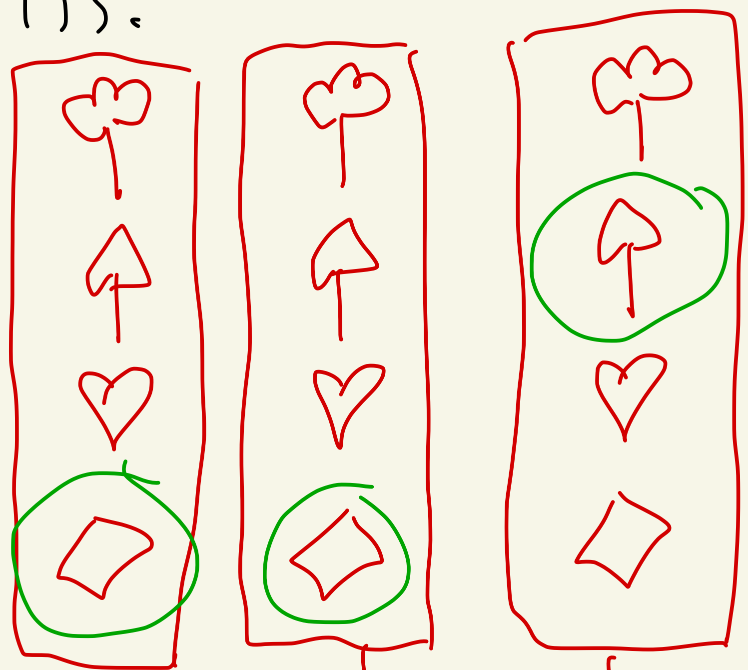
Ex: 2♣ 2♥ 8 5 K

Step 4: Fill in the 3 remaining suits.

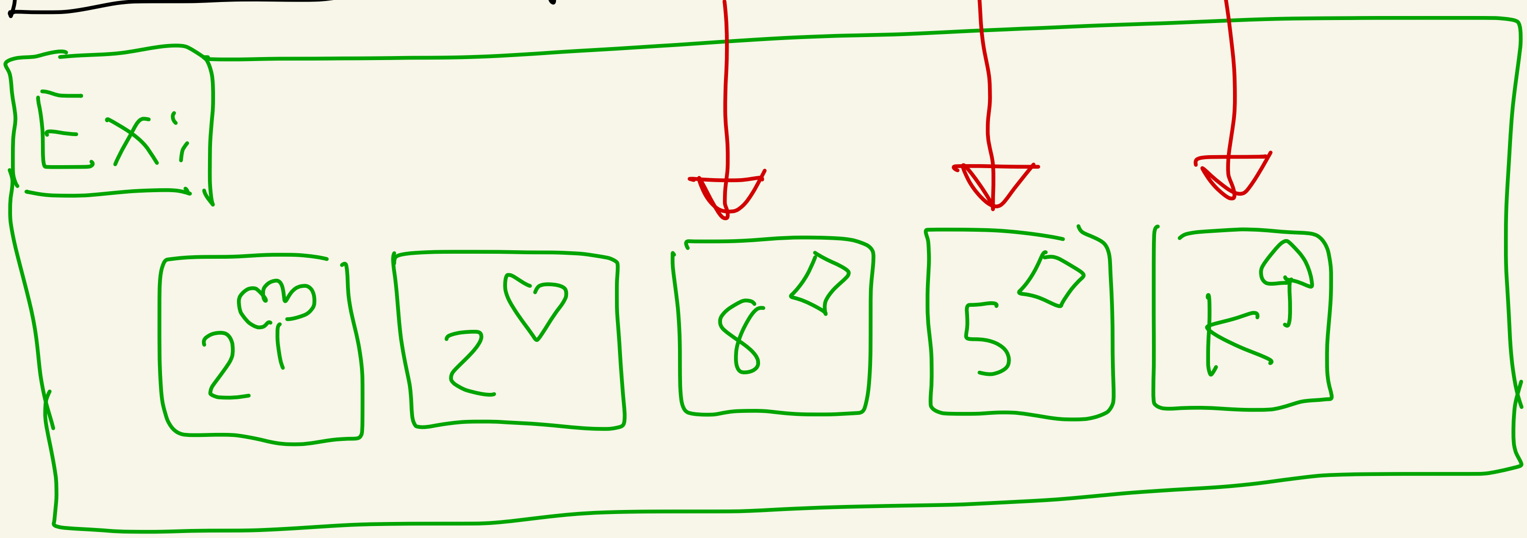
\# possibilities in step 4

$$= \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1}$$

$$= 4 \cdot 4 \cdot 4 = 64$$



Ex:



Thus, the total # of hands that are a pair and no better are

$$\underbrace{13}_{\text{step 1}} \cdot \underbrace{6}_{\text{step 2}} \cdot \underbrace{220}_{\text{step 3}} \cdot \underbrace{64}_{\text{step 4}}$$

$$= \boxed{1{,}098{,}240}$$

So the probability is

$$\frac{1{,}098{,}240}{2{,}598{,}960} \approx 0.422569\ldots$$

$$\approx \boxed{42\%}$$

Compound probabilities

How do we make a probability function when you do two experiments in a row where the outcome of the first experiment does not influence the outcome of the second experiment?

Ex: Suppose you flip a coin and then roll a 4-sided die. Let's make a probability space for this. [We use a normal coin & die]

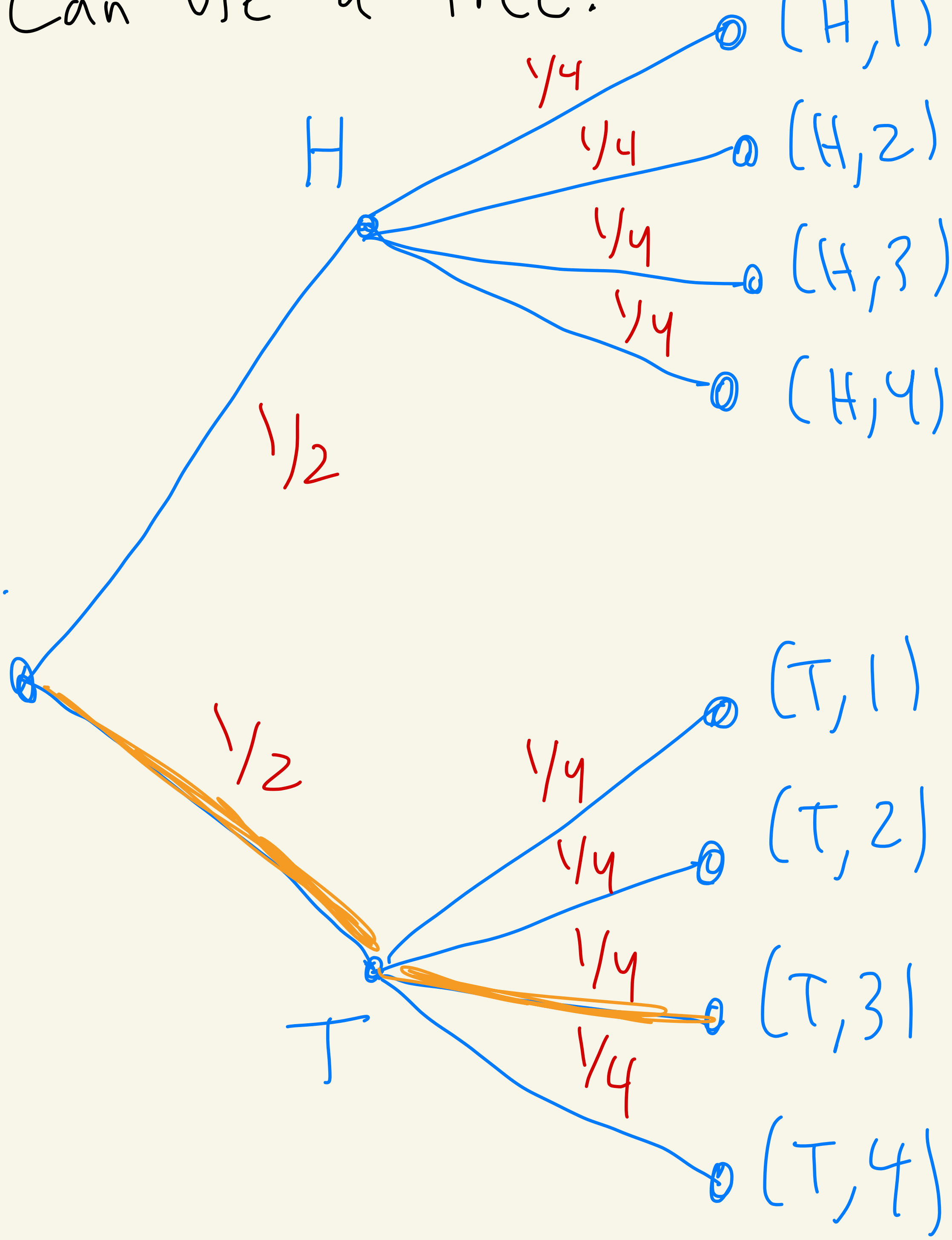
Sample space:

$$S = \{(H,1), (H,2), (H,3), (H,4), (T,1), (T,2), (T,3), (T,4)\}$$

$$= \underbrace{\{H, T\}}_{\text{sample space of flipping coin}} \times \underbrace{\{1,2,3,4\}}_{\text{sample space of rolling 4-sided die}}$$

Ω = set of all subsets of S

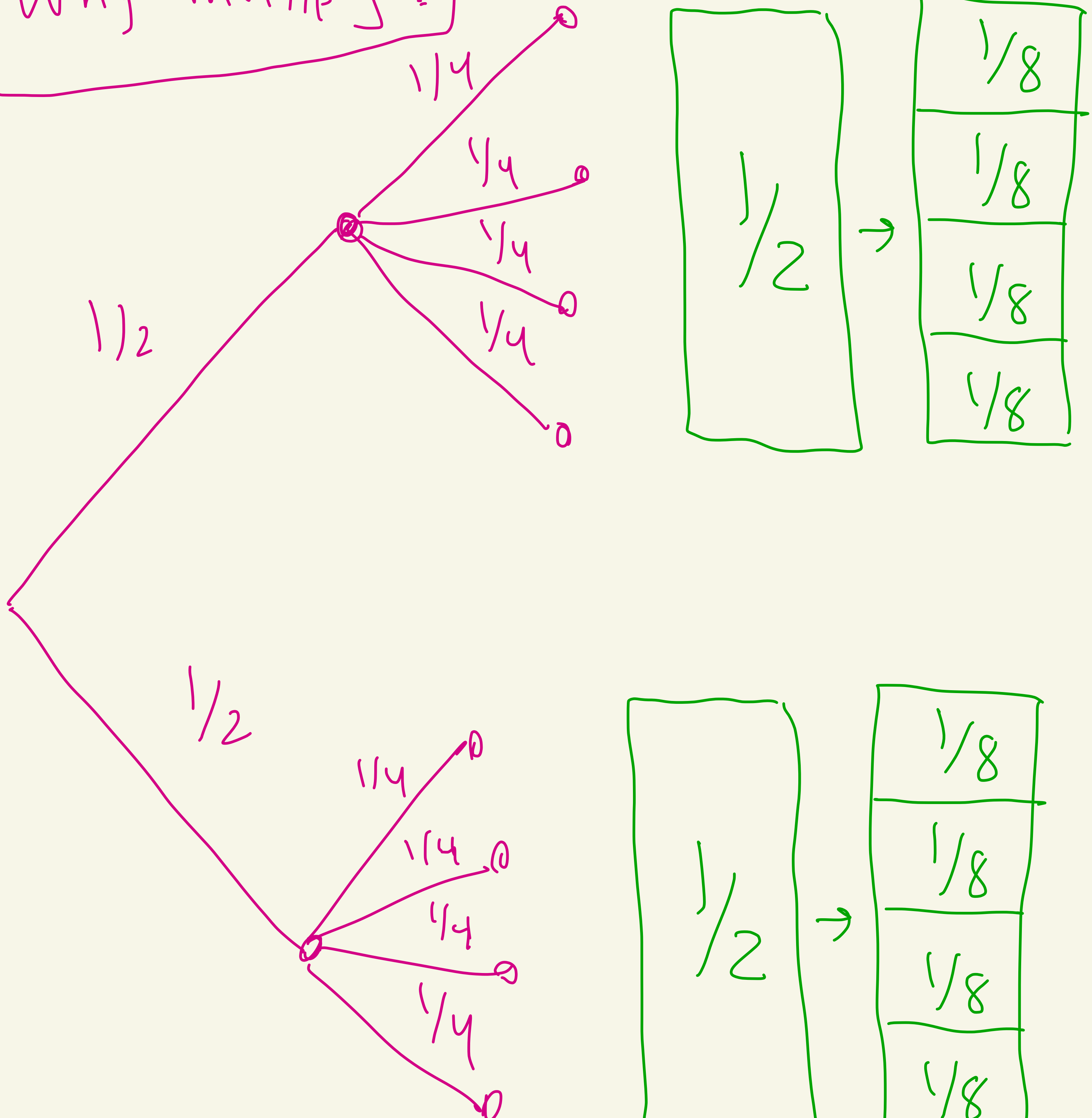
Let's make the probability function.
Can use a tree.



$$P((T,3)) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

multiply probabilities along the path to (T,3)

Why multiply?



How to do this in general

Suppose we want to do two experiments one after the other and the outcome of each experiment doesn't influence the outcome of the other.

Let (S_1, Ω_1, P_1) and (S_2, Ω_2, P_2) be probability spaces corresponding the first and second experiments.

Define (S, Ω, P) where

$$S = S_1 \times S_2$$

and

Ω is the smallest σ-algebra containing all subsets of S of the form $E_1 \times E_2$ where $E_1 \in \Omega_1$ and $E_2 \in \Omega_2$.

Define P on $S = S_1 \times S_2$ as follows:

$$P(\{(w_1, w_2)\}) = P_1(\{w_1\}) \cdot P_2(\{w_2\})$$

where $w_1 \in S_1$ and $w_2 \in S_2$.

If S is finite and E_1 is an event from Ω_1 and E_2 is an event from Ω_2 then

$$P(E_1 \times E_2) = \sum_{(e_1, e_2) \in E_1 \times E_2} P(\{(e_1, e_2)\})$$

$$= \sum_{(e_1, e_2) \in E_1 \times E_2} P_1(\{e_1\}) \cdot P_2(\{e_2\})$$

$$= \sum_{e_1 \in E_1} \sum_{e_2 \in E_2} P_1(\{e_1\}) \cdot P_2(\{e_2\})$$

$$= \left(\sum_{e_1 \in E_1} P_1(\{e_1\}) \right) \cdot \left(\sum_{e_2 \in E_2} P_2(\{e_2\}) \right)$$

$$= P_1(E_1) \cdot P_2(E_2)$$

Thus,

$$\begin{aligned} P(S) &= P(S_1 \times S_2) \\ &= P_1(S_1) \cdot P_2(S_2) \\ &= 1 \cdot 1 \\ &= 1 \end{aligned}$$

Ex: Suppose you have a 4-sided weighted die labeled 1, 2, 3, 4. From rolling the die lots of times you have determined the probabilities are:

# on die	1	2	3	4
probability	$1/8$	$1/4$	$1/2$	$1/8$

Let's model first rolling this weighted die and then flipping a fair coin.

first experiment

$$S_1 = \{1, 2, 3, 4\}$$

Ω_1 = all subsets of S_1

$$P_1(\{1\}) = 1/8$$
$$P_1(\{2\}) = 1/4$$
$$P_1(\{3\}) = 1/2$$
$$P_1(\{4\}) = 1/8$$

second experiment

$$S_2 = \{H, T\}$$

Ω_2 = all subsets of S_2

$$P_2(\{H\}) = 1/2$$
$$P_2(\{T\}) = 1/2$$

probability space of rolling die then flipping coin

$$S = S_1 \times S_2 = \{(1,H), (2,H), (3,H), (4,H), (1,T), (2,T), (3,T), (4,T)\}$$

Ω = all subsets of S

$$P(\{(1,H)\}) = P_1(\{1\}) \cdot P_2(\{H\})$$
$$= \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16}$$

$$P(\{(2,H)\}) = P_1(\{2\}) \cdot P_2(\{H\})$$
$$= \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

$$P(\{(3,H)\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

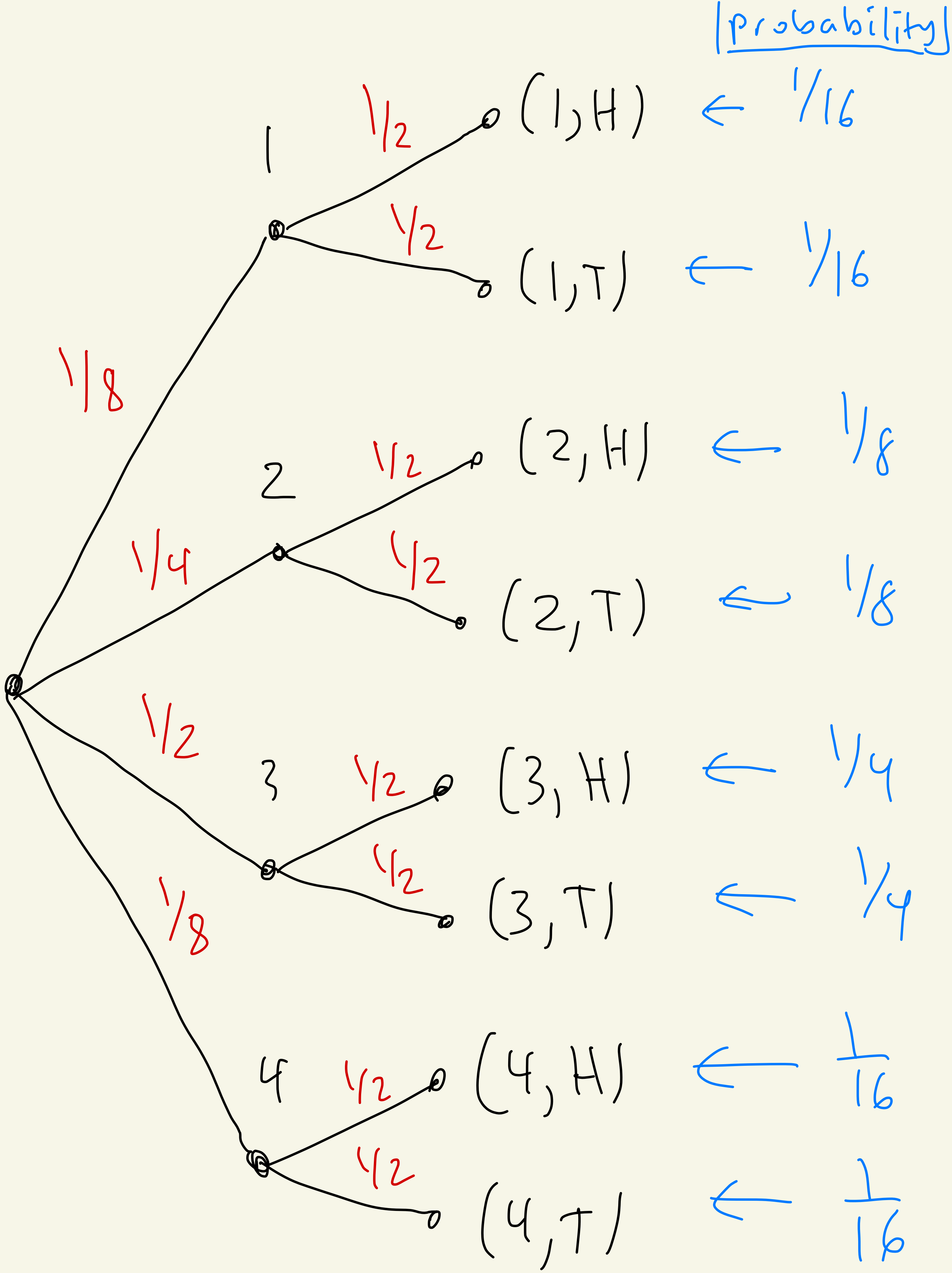
$$P(\{(4,H)\}) = \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16}$$

$$P(\{(1,T)\}) = \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16}$$

$$P(\{(2,T)\}) = 1/4 \cdot \frac{1}{2} = \frac{1}{8}$$

$$P(\{(3,T)\}) = 1/2 \cdot \frac{1}{2} = \frac{1}{4}$$

$$P(\{(4,T)\}) = 1/8 \cdot \frac{1}{2} = \frac{1}{16}$$


probability
1
2
3
4
1/8
1/4
1/2
1/8
1/2
(1, H) ← 1/16
(1, T) ← 1/16
(2, H) ← 1/8
(2, T) ← 1/8
(3, H) ← 1/4
(3, T) ← 1/4
(4, H) ← 1/16
(4, T) ← 1/16